EVERY CYCLE WITH CHORD HAMILTONIAN PATH IS HARMONIOUS AND ELEGANT

A. Anand Ephremnath\textsuperscript{a}, A. Elumalai\textsuperscript{b}

Address for Correspondence
\textsuperscript{a}Department of Mathematics, Surya Group of Institutions, School of Engineering and Technology, Vikiravandi, Villupuram - 605652, India
\textsuperscript{b}Department of Mathematics, Valliammai Engineering College, Kattankulathur- 603203, India

ABSTRACT
A graph $G$ is called a cycle with chord Hamiltonian path, if $G$ is obtained from the cycle $C_n: v_0, v_1, v_2, \ldots, v_{n-1}, v_n$, for all $n \geq 6$ by adding the chords $v_1v_{n-1}, v_0v_2, v_2v_{n-2}, \ldots, v_nv_\beta$. In this paper we prove that every cycle $C_n (n \geq 6)$ with chord Hamiltonian path is Harmonious and Elegant.

KEYWORDS: Graph labeling; Elegant labeling; Harmonious labeling; Chord Hamiltonian path.

Subject Classification Code: 05C78

1. INTRODUCTION
In 1981, Chang, Hsu, and Rogers\textsuperscript{[2]} defined an elegant labeling $f$ of a graph $G$ with $q$ edges as an injective function from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that when each edge $xy$ is assigned the label $f(x)+f(y) \pmod{(q+1)}$ the resulting edge labels are distinct and nonzero. Graham and Sloane\textsuperscript{[9]} introduced harmonious labeling $f$ of a graph $G$ with $q$ edges as an injective function from the vertices of $G$ to the set $\{0, 1, \ldots, q-1\}$ such that when each edge $xy$ is assigned the label $f(x)+f(y) \pmod{q}$ the resulting edge labels are distinct and they proved that caterpillars are harmonious. Aldred and McKay\textsuperscript{[1]} used a computer to show that all trees with at most 26 vertices are harmonious. Gnanajothi\textsuperscript{[8]} has shown that webs with odd cycles are harmonious. Graham and Sloane\textsuperscript{[9]} proved that $C_n$ is harmonious if and only if $n$ is odd. Chang et al.\textsuperscript{[2]} proved that $C_n$ is elegant when $n \equiv 0 \pmod{2}$ or $3 \pmod{4}$ and not elegant when $n \equiv 1 \pmod{4}$. Deb and Limaye\textsuperscript{[3]} have shown that triangular snakes are elegant if and only if the number of triangles is not equal to 3 \pmod{4}. A cycle with a chord is a cycle $C_n$ with any two non-adjacent vertices joined. Elumalai and Sethuraman\textsuperscript{[5,6]} defined a cycle with parallel $P_2$-chords as a graph obtained from a cycle $C_n (n \geq 6)$ and they proved that every cycle $C_n (n \geq 6)$ with parallel $P_2$-chords is graceful for $k = 3, 4, 6, 8 & 10$ and they conjecture that the cycle $C_n$ with parallel $P_2$-chords is graceful for all even $k$, and they have also proved that for $n \geq 6$, every $n$-cycle with parallel chords is graceful. Elumalai and Anand Ephremnath\textsuperscript{[4]} defined cycle with zigzag chords and proved that for $n \geq 8$, every $n$-cycle with zigzag chords is graceful.

For detail survey on graph labeling in the field of Elegant and Harmonious labeling one can refer to Gallian\textsuperscript{[7]}.

In this paper we define a cycle with chord Hamiltonian path and prove that every cycle $C_n (n \geq 6)$ with chord Hamiltonian path is Harmonious and Elegant.

2. Main Results
A graph $G$ is called a cycle with chord Hamiltonian path, if $G$ is obtained from the cycle $C_n: v_0, v_1, v_2, \ldots, v_{n-1}, v_n$, for all $n \geq 6$ by adding the chords $v_1v_{n-1}, v_0v_2, v_2v_{n-2}, \ldots, v_nv_\beta$, where

\begin{align*}
a & = \frac{n-2}{2} \quad \text{and} \quad \beta = \frac{n+2}{2} \quad \text{if} \quad n = 0, 2, 4 \pmod{6} \\
b & = \frac{n+3}{2} \quad \text{and} \quad \beta = \frac{n-1}{2} \quad \text{if} \quad n = 1, 3, 5 \pmod{6}
\end{align*}

The graphs are described below

![Diagram](image1)

**Figure 1.** (a) if $n = 0, 2, 4 \pmod{6}$; (b) if $n = 1, 3, 5 \pmod{6}$

**Theorem 2.1:** For $n \geq 6$, every cycle $C_n$ with chord Hamiltonian path is Harmonious.

**Proof:** Let $G$ be a cycle $C_n (n \geq 6)$ with chord Hamiltonian path, then $G$ can be described as in Figure 1. Let $M$ be the number of edges of $G$. To prove $G$ is Harmonious, it is sufficient to prove it in the following six cases,
1) \( n \equiv 0 \pmod{6}, \ n \geq 6 \)
2) \( n \equiv 1 \pmod{6}, \ n \geq 6 \)
3) \( n \equiv 2 \pmod{6}, \ n \geq 6 \)
4) \( n \equiv 3 \pmod{6}, \ n \geq 6 \)
5) \( n \equiv 4 \pmod{6}, \ n \geq 6 \)
6) \( n \equiv 5 \pmod{6}, \ n \geq 6 \)

**Case 1** When \( n \equiv 0 \pmod{6}, \ n \geq 6 \), let \( n = 6k \) \((k \geq 1)\) then \( M = 2n - 3 \).

(i) When \( k = 1 \) we have \( n = 6 \) and \( M = 9 \), then define \( f(v_0) = 1, \ f(v_1) = 8, \ f(v_2) = 7, \ f(v_3) = 6, \ f(v_4) = 5 \), \( f(v_5) = 0 \).

(ii) When \( n = 6k \) \((k \geq 2)\) then \( M = 2n - 3 \)
Define
\[
f(v_0) = 1, \ f(v_{n-1}) = 0, \ f\left(\frac{v_i}{2}\right) = n - 1
\]
\[
f(v_{3i+j-3}) = M - 6(i - 1) - j, 1 \leq i \leq \frac{n}{6}, 1 \leq j \leq 3
\]
\[
f(v_{n+6i+2j}) = n + 6i + j - 4, 1 \leq i \leq \frac{n-6}{6}, 1 \leq j \leq 3
\]

**Case 2** When \( n \equiv 1 \pmod{6}, \ n \geq 6 \), let \( n = 6k + 1 \) \((k \geq 1)\) then \( M = 2n - 3 \).

(i) When \( k = 1 \) we have \( n = 7 \) and \( M = 11 \), then define \( f(v_0) = 1, \ f(v_1) = 10, \ f(v_2) = 9, \ f(v_3) = 8, \ f(v_4) = 6, \ f(v_5) = 7, \ f(v_6) = 0 \).

(ii) When \( n = 6k + 1 \) \((k \geq 2)\) then \( M = 2n - 3 \)
Define
\[
f(v_0) = 1, \ f(v_{n-1}) = 0, \ f\left(\frac{v_i}{2}\right) = n - 1, \ f\left(\frac{v_i}{3}\right) = n
\]
\[
f(v_{3i+j-3}) = M - 6(i - 1) - j, 1 \leq i \leq \frac{n-1}{6}, 1 \leq j \leq 3
\]
\[
f(v_{n+6i+2j-2}) = n + 6i + j - 2, 1 \leq i \leq \frac{n-8}{6}, 1 \leq j \leq 3
\]

**Case 3** When \( n \equiv 2 \pmod{6}, \ n \geq 6 \), let \( n = 6k + 2 \) \((k \geq 1)\) then \( M = 2n - 3 \).

(i) When \( k = 1 \) we have \( n = 8 \) and \( M = 13 \), then define \( f(v_0) = 1, \ f(v_1) = 12, \ f(v_2) = 11, \ f(v_3) = 10, \ f(v_4) = 8, \ f(v_5) = 7, \ f(v_6) = 9, \ f(v_7) = 0 \).

(ii) When \( n = 6k + 2 \) \((k \geq 2)\) then \( M = 2n - 3 \)
Define
\[
f(v_0) = 1, \ f(v_{n-1}) = 0, \ f\left(\frac{v_i}{3}\right) = n, \ f\left(\frac{v_i}{4}\right) = n - 1, \ f\left(\frac{v_i}{5}\right) = n + 1
\]
\[
f(v_{3i+j-3}) = M - 6(i - 1) - j, 1 \leq i \leq \frac{n-2}{6}, 1 \leq j \leq 3
\]
\[
f(v_{n+6i+2j-2}) = n + 6i + j - 2, 1 \leq i \leq \frac{n-8}{6}, 1 \leq j \leq 3
\]

**Case 4** When \( n \equiv 3 \pmod{6}, \ n \geq 6 \), let \( n = 6k + 3 \) \((k \geq 1)\) then \( M = 2n - 3 \).

Define
\[
f(v_0) = 1, \ f(v_{n-1}) = 0, \ f\left(\frac{v_i}{2}\right) = n - 1
\]
\[
f(v_{3i+j-3}) = M - 6(i - 1) - j, 1 \leq i \leq \frac{n-3}{6}, 1 \leq j \leq 3
\]
\[
f(v_{n+6i+2j-2}) = n + 6i + j - 7, 1 \leq i \leq \frac{n-3}{6}, 1 \leq j \leq 3
\]

**Case 5** When \( n \equiv 4 \pmod{6}, \ n \geq 6 \), let \( n = 6k + 4 \) \((k \geq 1)\) then \( M = 2n - 3 \).

Define
\[
f(v_0) = 1, \ f(v_{n-1}) = 0, \ f\left(\frac{v_i}{2}\right) = n, \ f\left(\frac{v_i}{3}\right) = n - 1
\]
\[
f(v_{3i+j-3}) = M - 6(i - 1) - j, 1 \leq i \leq \frac{n-4}{6}, 1 \leq j \leq 3
\]
\[
f(v_{n+6i+2j-2}) = n + 6i + j - 6, 1 \leq i \leq \frac{n-4}{6}, 1 \leq j \leq 3
\]
Case : 6 When \( n = 5 \pmod{6}, n \geq 6 \), let \( n = 6k + 5 \ (k \geq 1) \) then \( M = 2n - 3 \).
Define
\[
f(v_0) = 1, \ f(v_{n-1}) = 0, \ f(v_{\frac{n}{2}+j}) = n+1, \ f(v_{\frac{n}{2}}) = n-1, \ f(v_{\frac{n}{2}+\frac{j}{2}}) = n
\]
\[
f(v_{3i+j-3}) = M - 6(i - 1) - j, 1 \leq i \leq \frac{n-5}{6}, 1 \leq j \leq 3
\]
\[
f(v_{3i+2j-5}) = n + 6i + j - 5, 1 \leq i \leq \frac{n-5}{6}, 1 \leq j \leq 3
\]
From the above vertex labeling in all the cases we observe that the function \( f \) of the graph \( G \) with \( M \) edges is an injective function from the vertices of \( G \) to the set \( \{0, 1, \ldots, M - 1\} \), such that when each edge \( xy \) is assigned the label \( f(x) + f(y) \pmod{M} \) the resulting edge labels are distinct from 0 to \( M - 1 \). So \( G \) is Harmonious.

**Theorem 2.2:** For \( n \geq 6 \), every cycle \( C_n \) with chord Hamiltonian path is Elegant.

**Proof:**
Let \( G \) be a cycle \( C_n \) with chord Hamiltonian path, then \( G \) can be described as in Figure 1. Let \( M \) be the number of edges of \( G \). To prove \( G \) is Elegant, it is sufficient to prove it in the following six cases,
1) \( n \equiv 0 \pmod{6}, n \geq 6 \)
2) \( n \equiv 1 \pmod{6}, n \geq 6 \)
3) \( n \equiv 2 \pmod{6}, n \geq 6 \)
4) \( n \equiv 3 \pmod{6}, n \geq 6 \)
5) \( n \equiv 4 \pmod{6}, n \geq 6 \)
6) \( n \equiv 5 \pmod{6}, n \geq 6 \)

**Case: 1** When \( n \equiv 0 \pmod{6}, n \geq 6 \), let \( n = 6k \ (k \geq 1) \) then \( M = 2n - 3 \).
(i) When \( k = 1 \) we have \( n = 6 \) and \( M = 9 \), then define \( f(v_0) = 2, \ f(v_1) = 9, \ f(v_2) = 8, \ f(v_3) = 7, \ f(v_4) = 6, \ f(v_5) = 0 \).
(ii) When \( n = 6k \ (k \geq 2) \) then \( M = 2n - 3 \)
Define
\[
f(v_0) = 2, \ f(v_{n-1}) = 0, \ f(v_{\frac{n}{2}+\frac{j}{2}}) = n
\]
\[
f(v_{3i+2j-5}) = M - 6(i - 1) - j, 1 \leq i \leq \frac{n}{6}, 1 \leq j \leq 3
\]
\[
f(v_{3n+2j-5}) = n + 6i + j - 5, 1 \leq i \leq \frac{n-5}{6}, 1 \leq j \leq 3
\]

**Case : 2** When \( n \equiv 1 \pmod{6}, n \geq 6 \), let \( n = 6k + 1 \ (k \geq 1) \) then \( M = 2n - 3 \).
(i) When \( k = 1 \) we have \( n = 7 \) and \( M = 11 \), then define \( f(v_0) = 2, \ f(v_1) = 11, \ f(v_2) = 10, \ f(v_3) = 9, \ f(v_4) = 7, \ f(v_5) = 8, \ f(v_6) = 0 \).
(ii) When \( n = 6k + 1 \ (k \geq 2) \) then \( M = 2n - 3 \)
Define
\[
f(v_0) = 2, \ f(v_{n-1}) = 0, \ f(v_{\frac{n}{2}+\frac{j}{2}}) = n
\]
\[
f(v_{3i+2j-5}) = M - 6(i - 1) - j, 1 \leq i \leq \frac{n-1}{6}, 1 \leq j \leq 3
\]
\[
f(v_{3n+2j-5}) = n + 6i + j - 5, 1 \leq i \leq \frac{n-7}{6}, 1 \leq j \leq 3
\]

**Case : 3** When \( n \equiv 2 \pmod{6}, n \geq 6 \), let \( n = 6k + 2 \ (k \geq 1) \) then \( M = 2n - 3 \).
(i) When \( k = 1 \) we have \( n = 8 \) and \( M = 13 \), then define \( f(v_0) = 2, \ f(v_1) = 13, \ f(v_2) = 12, \ f(v_3) = 11, \ f(v_4) = 9, \ f(v_5) = 8, \ f(v_6) = 10, \ f(v_7) = 0 \).
(ii) When \( n = 6k + 2 \ (k \geq 2) \) then \( M = 2n - 3 \)
Define
\[
f(v_0) = 2, \ f(v_{n-1}) = 0, \ f(v_{\frac{n}{2}+\frac{j}{2}}) = n
\]
\[
f(v_{3i+2j-5}) = M - 6(i - 1) - j, 1 \leq i \leq \frac{n-2}{6}, 1 \leq j \leq 3
\]
\[
f(v_{3n+2j-5}) = n + 6i + j - 5, 1 \leq i \leq \frac{n-8}{6}, 1 \leq j \leq 3
\]

**Case : 4** When \( n \equiv 3 \pmod{6}, n \geq 6 \), let \( n = 6k + 3 \ (k \geq 1) \) then \( M = 2n - 3 \).
Define  
\[ f(v_0) = 2, \ f(v_{n-1}) = 0, \ f(v_i) = n \]
\[ f(v_{2i+2j-3}) = M - 6(i - 1) - j + 1, 1 \leq i \leq \frac{n-3}{6}, 1 \leq j \leq 3 \]
\[ f(v_{n+6i+2j-7}) = n + 6i + j - 6, 1 \leq i \leq \frac{n-3}{6}, 1 \leq j \leq 3 \]

**Case 5**  When \( n = 4 \pmod{6} \), \( n \geq 6 \), let \( n = 6k + 4 \ (k \geq 1) \) then \( M = 2n - 3 \).

Define  
\[ f(v_0) = 2, \ f(v_{n-1}) = 0, \ f(v_i) = n + 1, \ f(v_j) = n \]
\[ f(v_{3i+3j-3}) = M - 6(i - 1) - j + 1, 1 \leq i \leq \frac{n-4}{6}, 1 \leq j \leq 3 \]
\[ f(v_{n+6i+2j-6}) = n + 6i + j - 5, 1 \leq i \leq \frac{n-4}{6}, 1 \leq j \leq 3 \]

**Case 6**  When \( n = 5 \pmod{6} \), \( n \geq 6 \), let \( n = 6k + 5 \ (k \geq 1) \) then \( M = 2n - 3 \).

Define  
\[ f(v_0) = 2, \ f(v_{n-1}) = 0, \ f(v_i) = n + 2, \ f(v_j) = n \]
\[ f(v_{3i+3j-3}) = M - 6(i - 1) - j + 1, 1 \leq i \leq \frac{n-5}{6}, 1 \leq j \leq 3 \]
\[ f(v_{n+6i+2j-5}) = n + 6i + j - 4, 1 \leq i \leq \frac{n-5}{6}, 1 \leq j \leq 3 \]

From the above vertex labeling in all the cases we observe that the function \( f \) of the graph \( G \) with \( M \) edges is an injective function from the vertices of \( G \) to the set \( \{0, 1, \ldots, M\} \), such that when each edge \( xy \) is assigned the label \( f(x) + f(y) \pmod{(M+1)} \) the resulting edge labels are distinct and non zero from 1 to \( M \). So \( G \) is Elegant.

**Examples:**

**Figure 2. Harmonious labeling for \( n=10, M=17 \)**

**Figure 3. Elegant labeling for \( n=11, M=19 \)**

**CONCLUSION**

In this paper we defined chord Hamiltonian path on a cycle \( C_n (n \geq 6) \) as shown in Figure 1, furthermore in theorem 2.1, we have shown that every cycle \( C_n (n \geq 6) \) with chord Hamiltonian path is Harmonious and also in theorem 2.2 we have shown that every cycle \( C_n (n \geq 6) \) with chord Hamiltonian path is Elegant.

**REFERENCES**


*Int J Adv Engg Tech/Vol. VII/Issue II/April-June,2016/01-04*