Research Paper

HARMONIOUS LABELING OF CERTAIN GRAPHS

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ABSTRACT

Harmonious labeling of graph is getting lots of application in social networking, rare probability event and many more. Here we will discuss about some harmonious labeling techniques and some important theorems and examples based on those theorems.

KEYWORDS: Fan, Friendship graph, Harmonious labeling, Helm, Labeling of Graph, Ladders Peterson Graph, Wheel.

Subject Classification Code: 05C78

1. INTRODUCTION

Lots of research work is been carried out in the labeling of graphs in past few work since the first initiated by A. Rosa [6]. Due to development of information technology and solving typical algorithms of coloring problems – graph labeling is extremely useful.

2. DEFINITIONS

2.1. Fan

The fan fn ($n \geq 2$) is obtained by joining all vertices of P_n (Path of n vertices) to a further vertex called the center and contains n+ 1 vertex and 2n-1 edges. i.e. $f_n = P_n + K_1$.

Fan f4 is shown in the following Fig. I

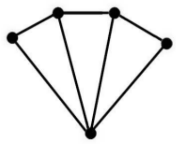

Figure I

2.2. Friendship Graph

A friendship graph F_n is a graph which consists of n triangles with a common vertex. Friendship graph F_4 is shown in the following Fig. II.

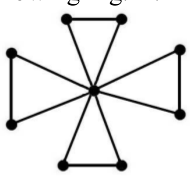

Fig. II.

2.3. Wheel

The wheel graph W_n is defined to be the join of $K_1 + C_n$ i.e. the wheel graph consists of edges which join a vertex of K_1 to every vertex of C_n. Fig. III shows a wheel W_3.

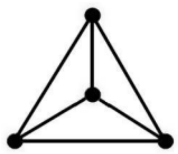

Figure III

2.4. Helm

The helm H_n is a graph obtained from a wheel by attaching a pendant vertex at each vertex of the n – cycle as shown in the Fig. IV

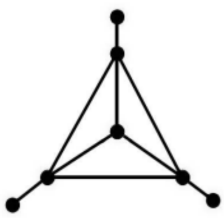

Figure IV

2.5. Ladders

The ladder Ln ($n \geq 2$) is the product graph P_2 x P_n which contains 2n vertices and 3n – 2 edges. The following

Fig. V shows a ladder L_4

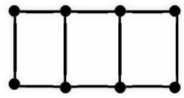

Figure V

2.6. Peterson Graph

It is a graph with 10 vertices and 15 edges. It is most commonly drawn as pentagon with a pentagram inside with five spokes as shown in the below Fig. VI.

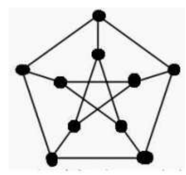

Figure VI

2.7. Labeling of Graph

If the vertices of the graph are assigned values subject to certain conditions than it is known as graph labeling.

Most of the graph labeling problem will have following three common characteristics

- A set of numbers from which vertex labels are chosen,
- A rule that assigns a value to each edge,
- A condition that these values must satisfy.

2.8. Harmonious Labeling

Let G be a graph with q edges. A function f is called harmonious labeling of graph G if f: V $\rightarrow$ {0, 1, 2,,,,q-1} is injective and the induced function f*: E $\rightarrow$ {0, 1, 2,,,,,,q} defined as f*(e = uv) = (f(u) + f(v))(mod q) is bijective. A Graph which admits harmonious labeling is called harmonious graph

3. PROPERTIES OF HARMONIOUS LABELING

3.1. Harmonious labeling is not unique.

Consider the following example where two different Harmonious labeling of the same graph are shown in Fig. VII

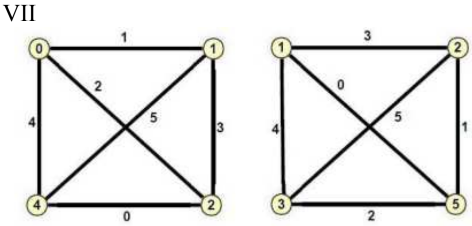

Figure VII

3.2. If f is a Harmonious labeling of any graph G with q edges, than af(x) + b is also harmonious labeling of G. where a is invertible element of $\mathbf{Z}$q and b is any arbitrary element of q. (Set of integers modulo q). We can verify this result from the following Fig. VIII (i) VIII (ii). In the Fig. VIII(i), the harmonious labeling of W4 is shown which corresponds to $\mathbb{Z}8$ = {0,1,2,3,4,5,6,7} Here 1, 3, 5 and 7 are invertible elements. We can label the same graph using the labeling of function 5f(x) + 4 as shown in Fig. VIII (ii), applying the result stated above.

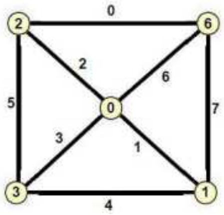

Figure VIII (i)

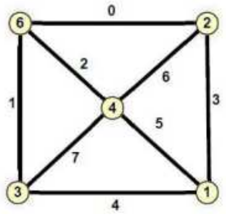

Figure VIII (ii)

3.3. Any vertex in a harmonious graph can be assigned the label 0.

3.4. Graham and Sloane [3] observed that in the case of trees exactly two vertices are assigned the same vertex label. Which can be verified from following Fig. IX.

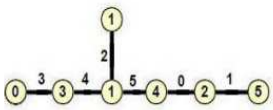

Fig IX (i)

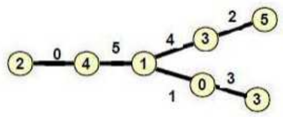

Fig IX (ii)

3.5. Graham and Sloane [3] conjectured that every tree is harmonious.

3.6. Aldred and Mckay[1] provided an algorithm and used computer to show that all trees with at most 26 vertices are harmonious.

3.7. Golomb[2] proved that complete graph is harmonious if and only if $n \leq 4$.

3.8. Graham and Sloane[3] proved that $K_{m,n}$ is harmonious if and only if m or n = 1. In the following fig. Harmonious labeling of $K_{1,7}$ is shown.

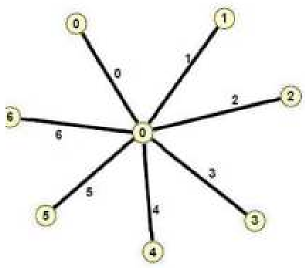

Figure X

3.9. Every graph with less than or equal to 5 vertices is harmonious except the following six.

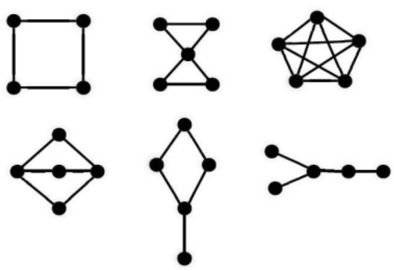

Figure XI

3.10. Let T be a harmonious labeled tree containing an edge e = wx labeled as f(w) + f(x) where x is a pendant vertex and f(x) is the repeated vertex label. If y is any other vertex in G, we may delete edge wx and vertex x and replace them with a new vertex z and edge e' = yz where z is labeled with f(z) = f(w) + f(x) – f(y). Above fact can be verified from the following Fig. XII(i) and XII(ii).

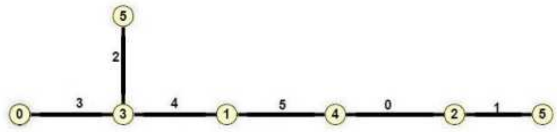

Figure XII (i)

Figure XII

3.11. The Peterson graph is Harmonious. The following Fig. XIII.

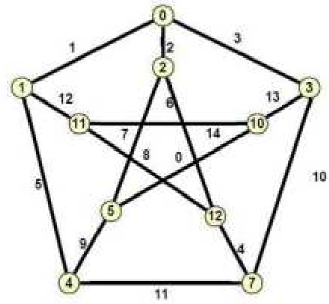

Figure XIII

3.12. Graham and Sloane[3], proved that wheel $W_n = C_n + K_1$ is harmonious. In the following Fig. XIV the Harmonious labeling of W_{12} is shown.

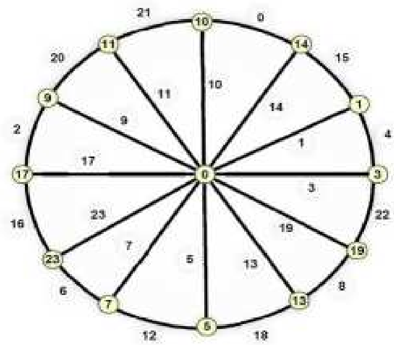

Figure XIV

3.13. $K_n^{(2)}$ is harmonious if n = 4 but not harmonious if n is odd or n = 6. In the following fig. XV harmonious labeling of $K_n^{(2)}$ is shown.

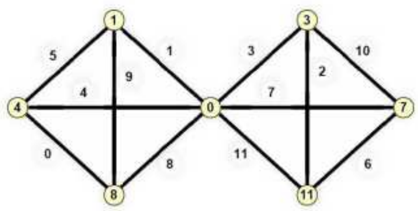

Figure XV

4. THEOREMS ON HARMONIOUS LABELING

4.1. The Cycle C_n ($n \geq 3$) is harmonious if and only if n is odd.

Proof: We shall consider the following two cases.

Case (1): If n is odd

Than we define the function f: V → {0,1,2,,,,n-1} by f(vi) = i - 1 by where i=1,2,,,,n than C_n admits a Harmonious labeling.

Case (2): If n is even

As n is even let n = 2m, and suppose $a_0, a_1, a_2,,,,,a_{2m-1}$ is harmonious labeling of cycle C_{2m}. The numbers a_0+a_1, $a_1+a_2,,,,,a_{2m-1}+a_0$ are congruent to modulo 2m. Adding these numbers we obtain $2s \equiv s$ (modulo 2m), where $s = 0+1+2+...+2m-1 \equiv m$ (modulo 2m). Hence $m \equiv 0$ (modulo 2m). This is a contradiction.

Therefore we conclude that the cycle C_n is harmonious if and only if $n \geq 3$ and n is odd.

4.2. All ladders except L_2 are harmonious.

Proof: It is obvious from above theorem 4.1 that $L_2 = C_4$ is not harmonious.

Now we consider the following two cases.

Case (1): n is odd

If n = 2a+1 (a ≥ 1) than label the vertices of one path by 0, a+1, 1, a+2, 2, a+3,,,and label the vertices of other path by 3a+1, 2a+1, 3a+2, 2a+2, 3a+3, 2a+3,,,,

Case (2): n is even

If n = 2a (a ≥ 1) than label the vertices of one path by 5a-3, 2a-1, 5a-1, 0, a+1, 1, a+2, 2,,,, 2a-2, a-2 and label the vertices of other path by 4a-1, 3a-2, 6a-2, 3a, 2a, 3a+1, 2a+1, ,,,,, 3a-2, 4a-2.

In the view of the above labeling pattern, the ladders L_n (n > 2) admits harmonious labeling.

4.3. Friendship graph F_n is harmonious except n ≡ 2(mod 4)

Proof: We consider the following three cases.

Case (1): If n ≡ 2 (mod 4)

Than F_2 is not harmonious according to a theorem. Since number of vertices is 5 and number of edges are 6. Which is not divisible by 4 or 8.

Case (2): If n ≡ 0 or 1 (mod 4)

than as proved by Skolem [], the numbers {0,1,2,,,,2n} may be partitioned into n pairs (a_r, b_r) with $b_r - a_r = r$ for r = 1,2,,,,n. than a harmonious labeling is obtained by labeling the vertices of the triangle with (0,r, n+a_r) for r = 1,2,,,n.

Case (3): If n ≡ 3 (mod 4)

Than {1,2,3,,,,2n-6} may be partitioned into n-3 pairs (a_r, b_r) with $b_r - a_r = r+2$ for r = 1,2,,,,n-3. we label the triangles of Fn with (0,1, 3n-1), (0,2, 3n-6), (0,3n-2, 3n-3) and (0,r+2, n+ a_r) for r = 1,2,,,n-3.

Thus, F_n is harmonious except n ≡ 2 (mod 4).

4.4. fn is harmonious.

Proof:

Let m = [n/2] and label the centre with 0 and the vertices of path with m, n, m+1, n+1, m+2,,,. Than we get the harmonious labeling of fan.

4.5. The graph g_n is harmonious (n ≥ 2)

Proof:

A harmonious labeling of g_{2m} is obtained by labeling the path with 2, 4, 8, 10, 14,,,,, 6m-4, 6m-2 and the two additional vertices with 0 and 1. But g_{2m+1}does not seem to have such a simple labeling.

5. EXAMPLES SUPPORTING ABOVE THEOREMS ON HARMONIOUS LABELING

5.1. In the following fig. XVI (i), XVI (ii), XVI (iii) a harmonious labeling of C_5, C_7 and C_9 is shown.

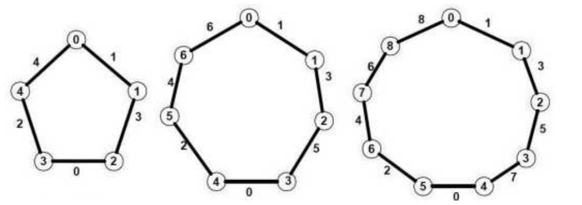

Figure XVI(i) Figure XVI(ii) Figure XVI(iii)

5.2. In the following fig. XVII harmonious labeling of L7 and L8 is shown where q = 19 and q = 22 edges respectively.

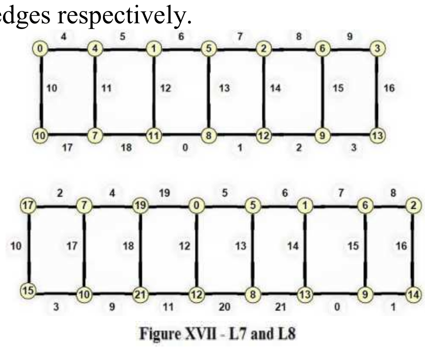

Figure XVII - L7 and L8

5.3. In the following fig. XVIII the harmonious labeling of F_4 and F_5 is shown.

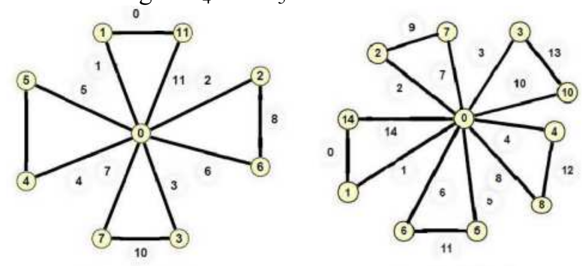

Figure XVIII F4 Figure XVIII F5

5.4. In the following fig. XIX the harmonious labeling of f_8 and f_9 is shown.

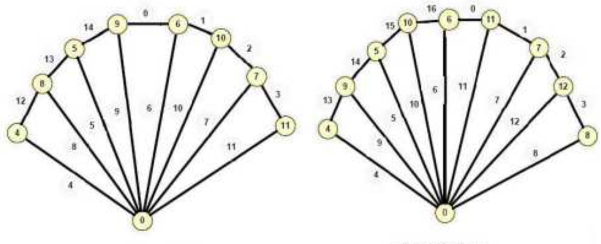

Figure XIX - f8 Figure XIX - f9

5.5. In the following fig. XX the harmonious labeling of g_8 is shown

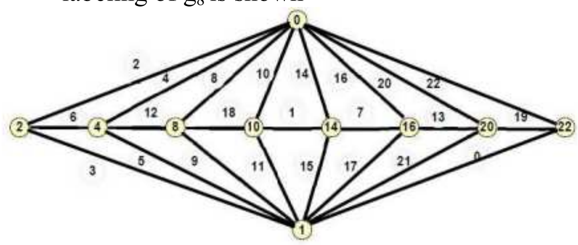

Figure XX

6. CONCLUSION

The harmonious labeling is one of the most important labeling techniques. As all the graphs are not harmonious, it is very interesting to investigate graphs or graph families which admit harmonious labeling. We have reported the harmonious labeling of various graphs

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